MATRIX CHAIN MULTIPLICATION

- **Matrix chain multiplication** is an optimization problem that can be solved using dynamic programming.
- Given a sequence of matrices, the goal is to find the most efficient way to multiply these matrices. It will merely decide the sequence of the matrix multiplications involved.
- There are many options to multiply a chain of matrices because matrix multiplication is associative. For example, if we had four matrices A, B, C, and D, can be done as

  (ABC)D = (AB)(CD) = A(BCD) .....
- Our aim is to find the order in which the matrices have to be parenthesized such that it requires minimum number of simple arithmetic operations to compute the product.
- For example if A is a 10 × 30 matrix, B is a 30 × 5 matrix, and C is a 5 × 60 matrix. Then,

  (AB)C = (10 * 30 * 5) + (10 * 5 * 60) = 1500 + 3000 = 4500 operations

  A(BC) = (30 * 5 * 60) + (10 * 30 * 60) = 9000 + 18000 = 27000 operations

  the first one requires less number of operations

Example:

The matrices have size 4 x 2, 2 x 3, 3 x 1, 1 x 5. Find the order in which the matrices have to be parenthesized such that it requires minimum number of simple arithmetic operations to compute the product.

Solution:

	1	2	3	4
1	0			
2		0		
3			0	
4				0

There are 4 matrices. So we have to take 4 X 4 table. Here

p_0 = 4 {No. of rows of 1^{st} matrix},

p_1= 2 {No. of columns of 1^{st} matrix/No. of rows of 2^{nd} matrix},

p_2 = 3 {No. of columns of 2^{nd} matrix/No. of rows of 3^{rd} matrix },

p_3 = 1 {No. of columns of 3^{rd} matrix/No. of rows of 4^{th} matrix },

p_4 = 5 {No. of columns of 4^{th} matrix}

<u>Calculation of Product of 2 matrices</u>:

1. $M[1,2] = M_1 \times M_2$

$= (4 \times 2) \times (2 \times 3)$

$= 4 \times 2 \times 3 = 24$

2. $M[2,3] = M_2 \times M_3$

$= (2x3) \times (3x1)$

$= 2 \times 3 \times 1 = 6$

3. $M[3,4] = M_3 \times M_4$

$= (3 \times 1) \times (1 \times 5)$

$= 3 \times 1 \times 5 = 15$

After filling the calculated values in the table

	1	2	3	4
1	0	24		
2		0	6	
3			0	15
4				0

<u>Calculation of Product of 3 matrices</u>:

To find M[1,3]

There are two cases by which we can solve this multiplication:

$(M_1 \times M_2) \times M_3$ or $M_1 \times (M_2 \times M_3)$

After solving both cases we choose the case which require minimum number of operations

$M[1,3] = \min\{(M[1,2] + M[3,3] + p_0 * p_2 * p_3), (M[1,1] + M[2,3] + p_0 * p_1 * p_3)\}$

$= \min\{(24 + 0 + 4 * 3 * 1), (0 + 6 + 4 * 2 * 1)\}$

$= \min\{36 , 14\}$

$= 14$

To find M[2,4]

There are two cases by which we can solve this multiplication:

$(M_2 \times M_3) \times M_4$ or $M_2 \times (M_3 \times M_4)$

After solving both cases we choose the case which require minimum number of operations

$M[2,4] = \min\{(M[2,3] + M[4,4] + p_1 * p_3 * p_4), (M[2,2] + M[3,4] + p_1 * p_2 * p_4)\}$

$= \min\{(6 + 0 + 2 * 1 * 5), (0 + 15 + 2 * 3 * 5)\}$

$= \min\{16 , 45\}$

$= 16$

Now the table becomes

	1	2	3	4
1	0	24	14	
2		0	6	16
3			0	15
4				0

<u>Calculation of Product of 4 matrices</u>:

To find M[1,4]

There are three cases by which we can solve this multiplication:

$(M_1 \times M_2 \times M_3) \times M_4$

$(M_1 \times M_2) \times (M_3 \times M_4)$

$M_1 \times (M_2 \times M_3 \times M_4)$

After solving all the cases we choose the case which require minimum number of operations

$M[1,4] = \min\{(M[1,3] + M[4,4] + p_0 * p_3 * p_4), (M[1,2] + M[3,4] + p_0 * p_2 * p_4),$

$(M[1,1] + M[2,4] + p_0 * p_1 * p_4) \}$

$= \min\{(14 + 0 + 4 * 1 * 5), (24 + 15 + 4 * 3 * 5), (0 + 16 + 4 * 2 * 5)\}$

$= \min\{34, 99, 56\}$

$= 34$

Now the table becomes

	1	2	3	4
1	0	24	14	34
2		0	6	16
3			0	15
4				0

The efficient way to multiply the given set of matrices is

$(M_1 \text{ x } M_2 \text{ x } M_3) \text{ x } M_4$

$= (M_1 \text{ x } (M_2 \text{ x } M_3)) \text{ x } M_4$

which requires 34 operations